

Grünwald version of van der Waerden's theorem for semi-modules

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Abstract

Let $(G, +)$ be any given semimodule over a discrete semiring $(R, +, \cdot)$ with a finite coloring, say $G = B_1 \cup \cdots \cup B_q$. By establishing a Regional Multiple Recurrence Theorem for semimodules, we prove that one of the colors j has the property that if $F \subseteq G$ is any finite set, then one can find some “syndetic” subset D_F of $(R, +)$ such that for each $d \in D_F$ there is some $a \in B_j$ with $a + dF \subseteq B_j$. This in turn implies that each Bohr almost point is multiply uniformly recurrent.

Keywords: Van der Waerden theorem · Multiple recurrence · Bohr almost periodicity and multiple uniform recurrence

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1. Introduction

1.1. Van der Waerden theorems

B. L. Van der Waerden's Theorem, conjectured by Baudet and proved in 1927, states (in one of several equivalent formulations) that if $\mathbb{N} = \{1, 2, 3, \dots\}$ is partitioned into finitely many sets, say $\mathbb{N} = B_1 \cup \cdots \cup B_q$, then one of these sets B_j contains arithmetic progressions of arbitrary finite length (cf. [25, 15]).

Since for any finite set $F \subset \mathbb{N}$ there is some $l \geq 1$ such that $F \subseteq \{1, 2, \dots, l\}$, hence van der Waerden's theorem is equivalent to the 1-dimensional case of Grünwald's Theorem:

Grünwald ([23, 15]). *Let $\mathbb{N}^m = B_1 \cup \cdots \cup B_q$ be an arbitrary finite partition of the m -dimensional positive lattice \mathbb{N}^m , where $1 \leq m < \infty$. Then one of the sets B_j has the property that if $F \subset \mathbb{N}^m$ is any finite set, then B_j contains a translate of a dilation of F : $a + bF \subset B_j$ where $a \in \mathbb{N}^m, b \in \mathbb{N}$.*

Many extensions of the above theorem have been made since Furstenberg 1981; see, e.g., [7, 5] for polynomial extensions of $b \in \mathbb{N}$. Another direction is for extensions of \mathbb{N}^m to semigroups.

Let $(G, +)$ be any nontrivial additive semigroup; then an analogue of van der Waerden's Theorem holds trivially by van der Waerden's Theorem itself. Indeed, let $g \in G$ be an arbitrary nonzero element, set $\widehat{\mathbb{N}} = \{ng : n \in \mathbb{N}\}$ where $ng = g + \cdots + g$ (n times), and define a homomorphism $\varphi: n \mapsto ng$ from \mathbb{N} onto $\widehat{\mathbb{N}}$. If $G = B_1 \cup \cdots \cup B_q$ is any finite partition of G , then $\widehat{\mathbb{N}} = \widehat{B}_1 \cup \cdots \cup \widehat{B}_q$, where $\widehat{B}_j = \{n : ng \in B_j\}$, is a finite partition of \mathbb{N} and we say that \widehat{B}_j contains $(l+1)$ -length arithmetic progressions $\{a, a+d, \dots, a+ld\}$ for any $l \geq 1$ by van der Waerden's

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Theorem. Via φ this implies that B_j contains arithmetic progressions $\{x, x + y, \dots, x + ly\}$ where $x = ag$ and $y = dg$ of every finite length $l + 1$ as well.

However to such a version of van der Waerden's theorem there is no guarantee in general that the "common difference" y of $\{x, x + y, \dots, x + ly\}$ will not be a zero element of the additive semigroup $(G, +)$. To avoid such triviality, using the Stone-Ćech compactification of discrete semigroup and ultrafilter methods, as a result of their Central Sets Theorem, V. Bergelson and N. Hinman in 1992 proved a strengthened version of van der Waerden's theorem as follows:

V. Bergelson and N. Hindman ([6, Corollary 3.2]). *Let $(G, +)$ be an abelian cancellative semigroup, let $\langle d_m \rangle_1^\infty$ be a sequence in G with $d_m \neq d_n$ for $m \neq n$, and let $G = B_1 \cup \dots \cup B_q$. Then there exists some B_j such that to any $l \in \mathbb{N}$, one can find $a \in G$ and $d \in FS(\langle d_m \rangle_1^\infty)$ with $d \neq 0$ such that $a, a + d, \dots, a + ld \in B_j$.*

Since every semigroup $(G, +)$ is just a semimodule over the semiring $(\mathbb{N}, +, \cdot)$ or $(\mathbb{Z}_+, +, \cdot)$, we will further generalize Gr unwald's version from \mathbb{Z} or \mathbb{N} to any general semimodules in the present paper.

Recall that by a *semiring* $(R, +, \cdot)$ it means a nonempty set R , together with two laws of composition called *addition* $+$ and *multiplication* \cdot respectively, satisfying the following axioms:

- $(R, +)$ is an *abelian* semigroup with the zero element 0 ;¹
- (R, \cdot) is a semigroup (not necessarily commutative) with a *unit* element 1 , which is *associative*: $(x \cdot y) \cdot z = x \cdot (y \cdot z) \ \forall x, y, z \in R$ and which is such that $0x = 0 \ \forall x \in R$;
- $(R, +, \cdot)$ is *distributive*:

$$(x + y) \cdot z = x \cdot z + y \cdot z \quad \text{and} \quad z \cdot (x + y) = z \cdot x + z \cdot y$$

for all $x, y, z \in R$.

Moreover, for convenience later on, we now introduce a technical condition:

(*) We shall say $(R, +, \cdot)$ is an $*$ -semiring if for any $s_1, \dots, s_k \in R$ where $1 \leq k < \infty$,

$$\mathfrak{N}_{s_1} \cup \dots \cup \mathfrak{N}_{s_k} \neq R, \quad \text{where } \mathfrak{N}_s := \{t \in R : s + t = 0\}.$$

It is easy to see that if $(R, +, \cdot)$ is an infinite ring, then it is an $*$ -ring automatically; moreover, if $(R, +)$ is cancellative infinite, then condition (*) holds.

Clearly $(\mathbb{Z}_+^n, +, \cdot)$ and $(\mathbb{R}_+^n, +, \cdot)$ are commutative $*$ -semirings with $0 = (0, \dots, 0)$, $1 = (1, \dots, 1)$ where

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

and

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = (x_1 y_1, \dots, x_n y_n)$$

for any $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}_+^n$. On the other hand, let $\mathbb{R}_+^{n \times n}$ be the set of all real $n \times n$ nonnegative matrices; then $(\mathbb{R}_+^{n \times n}, +, \circ)$ is an $*$ -semiring cancellative noncommutative.

¹If $(R, +, \cdot)$ itself does not have the zero element then we will adjoint 0 to it by letting $0 + t = t + 0 = t$ and $0 \cdot t = t \cdot 0 = 0$ for all $t \in R$, and further consider $(R \cup \{0\}, +, \cdot)$ instead of $(R, +, \cdot)$.

Basic Notion. Under the discrete topology, a subset N of the semiring $(R, +, \cdot)$ is called “syndetic” or “relatively dense” in $(R, +)$ if one can find a finite set $K \subseteq R$ such that

$$(K + t) \cap N \neq \emptyset \quad \forall t \in R.$$

See, e.g., [15, 9].

Clearly if N is a syndetic subset of an $*$ -semiring $(R, +, \cdot)$, then one can always find some element $d \in N$ with $d \neq 0$.

Basic Notion. Let $(R, +, \cdot)$ be a semiring. As usual a *semimodule over R* or an *R -semimodule* $(G, +)$ is an abelian semigroup with the zero element \mathbf{o} , usually written additively, together with a scalar multiplication $(t, g) \mapsto tg$ of R on G , such that, for all $r, t \in R$ and $g, h \in G$ we have the distributivity:

$$(r + t)g = rg + tg \quad \text{and} \quad r(g + h) = rg + rh;$$

and such that

$$1g = g \quad \text{and} \quad 0g = \mathbf{o} \quad \forall g \in G.$$

In a similar way, one defines a *right R -semimodule* via $(g, t) \mapsto gt$. We shall deal only with left R -semimodules, unless otherwise specified.

Clearly, $(\mathbb{R}^m, +)$ is an $(\mathbb{R}, +, \cdot)$ -module, $(\mathbb{Z}_+^m, +)$ is a $(\mathbb{Z}_+, +, \cdot)$ -semimodule, $(\mathbb{Z}_p^m, +)$ is a module over the p -adic integer ring $(\mathbb{Z}_p, +, \cdot)$, and $(\mathbb{Q}_p^m, +)$ is a module over the p -adic number field $(\mathbb{Q}_p, +, \cdot)$; cf. [21].

Here we will mainly prove the following more general generalization of Grünwald’s Theorem.

Theorem 1.1. *Let $G = B_1 \cup \dots \cup B_q$ be an arbitrary finite partition of a semimodule $(G, +)$ over a discrete semiring $(R, +, \cdot)$. Then one of the sets B_j has the property that if $F \subseteq G$ is any finite set, then one can find a syndetic subset D_F of $(R, +)$ such that for each $d \in D_F$ there is an $a \in B_j$ with $a + dF \subseteq B_j$.*

Note. If G is a right R -semimodule and we require $a + Fd \subseteq B_j$ instead of $a + dF \subseteq B_j$, then the statement holds as well. In addition, Theorem 1.1 has a “finitary formulation”; see Theorem 3.13 below.

This theorem is not subsumed by the above theorem of V. Bergelson and N. Hindman, because here every R -semimodule $(G, +)$ does not need to have a sequence $\langle d_m \rangle$ as in V. Bergelson and N. Hindman’s statement satisfying that for any finite subset F of G and some “syndetic” $d \in R$,

$$a + dF \subseteq \{a, a + d', \dots, a + ld'\}$$

for some $a \in B_j, d' \in FS(\langle d_m \rangle)$ and $l \geq 1$.

A direct consequence of Theorem 1.1 is the following theorem of van der Waerden type for some canonical modules:

Corollary 1. Let $G = \mathbb{R}^m$ (resp. $\mathbb{Q}^m, \mathbb{Z}_p^m, \mathbb{Q}_p^m$) and $G = B_1 \cup \dots \cup B_q$ be any finite partition of G , where \mathbb{R} is the real field, \mathbb{Q} the rational field, \mathbb{Z}_p the p -adic integer ring and \mathbb{Q}_p the p -adic number field. Then one of the sets B_j has the property that if F is a finite subset of G , then there are two elements $a \in B_j$ and $d \in R = \mathbb{R}$ (resp. $\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p$) with $d \neq 0$ such that $a + dF \subset B_j$.

More generally, from Theorem 1.1 follows

Corollary 2. Let $G = B_1 \cup \dots \cup B_q$ be any finite partition of a semigroup $(G, +)$. Then one of the sets B_j has the property that if F is a finite subset of G one can find a syndetic subset D_F of \mathbb{Z}_+ such that for each $d \in D_F$ there is an $a \in B_j$ with $a + dF \subseteq B_j$.

Following the framework of [4], to prove Theorem 1.1 we will need to prove a regional multiple recurrence theorem (cf. Theorem 1.2 below). Theorem 1.1 and Theorem 1.2 are in fact equivalent by the following

Corollary 3. Let $(G, +)$ be a semimodule over a discrete semiring $(R, +, \cdot)$, and let $\varphi: G \times X \rightarrow X$ be a discrete dynamical system on an arbitrary set X ; that is to say,

$$\varphi(\mathbf{o}, x) = x, \quad \varphi(f + g, x) = \varphi(f, \varphi(g, x)) \quad \forall f, g \in G \text{ and } x \in X.$$

Then for any finite partition of X , $X = X_1 \cup X_2 \cup \dots \cup X_q$, there exists a cell X_α such that for any $T_1, \dots, T_l \in G$ one can find a syndetic set D of $(R, +)$ with

$$X_\alpha \cap \varphi^{-dT_1}[X_\alpha] \cap \varphi^{-dT_l}[X_\alpha] \neq \emptyset \quad \forall d \in D.$$

Here $\varphi^{-g}[A] = \{x \in X: \varphi(g, x) \in A\}$ for any $g \in G$ and $A \subseteq X$.

Proof. Pick any point $x \in X$ and set $B_j = \{g \in G: \varphi(g, x) \in X_j\}$ for all $j = 1, 2, \dots, q$. Then $G = B_1 \cup \dots \cup B_q$ is a finite partition of G , and by Theorem 1.1 it follows that some cell B_α satisfies that to any finite set $F = \{\mathbf{o}, T_1, \dots, T_l\}$, there is a syndetic set D of $(R, +)$ so that for all $d \in D$, one can find some $a = a(d) \in G$ with $a + dF \subseteq B_\alpha$. This implies that

$$\varphi(a, x) \in X_\alpha \cap \varphi^{-dT_1}[X_\alpha] \cap \varphi^{-dT_l}[X_\alpha]$$

as desired. \square

The lacking implication “Theorem 1.2 \Rightarrow Theorem 1.1” will be obtained by using Furstenberg’s correspondence principle (cf. Proof of Theorem 1.1 in §3.5).

1.2. Topological multiple recurrence theorems

Let X be any topological space (not necessarily metrizable) and $(G, +)$ an abstract semimodule over a semiring $(R, +, \cdot)$ equipped with the discrete topology here; whenever the transformation $\varphi: G \times X \rightarrow X$ from the product space $G \times X$ to X is such that:

- $\varphi(\mathbf{o}, x) = x \quad \forall x \in X$,
- $\varphi(g, \cdot): X \rightarrow X$, for any $g \in G$, is continuous (sometimes we write $\varphi(g, \cdot) = \varphi^g(\cdot)$), and
- $\varphi(g + h, \cdot) = \varphi(g, \varphi(h, \cdot))$, i.e., $\varphi^{g+h} = \varphi^g \circ \varphi^h \quad \forall g, h \in G$;²

²Since we have required that $(G, +)$ is commutative, hence now $\varphi(g + h, \cdot) = \varphi(h + g, \cdot)$. However, when $(G, +)$ is a non-commutative discrete semigroup, then this is not the case.

then we shall call $\varphi: G \times X \rightarrow X$, sometimes written as $G \curvearrowright_\varphi X$, a *topological dynamical system* (t.d.s. for short) with the time-space $(G, +)$.

Given any point $x \in X$, $G_\varphi[x] = \varphi(G, x)$ is called the *orbit* of the motion $\varphi(\cdot, x)$. For any “time” $g \in G$, $\varphi^{-g}(\cdot): X \rightarrow X$ will be defined as the inverse (possibly multivalent) of the g -sample map $\varphi(g, \cdot): X \rightarrow X$.

A t.d.s. $G \curvearrowright_\varphi X$ is *minimal* if and only if there does not exist a nonempty proper closed subset Y of X such that $G_\varphi[Y] \subseteq Y$. Similarly one can define minimal subset of $G \curvearrowright_\varphi X$.

We can then obtain the following topological Regional Multiple Recurrence Theorem, which generalizes and strengthens a classical theorem of Furstenberg and Weiss 1978 [17, Theorem 1.5] that is for the invertible case of $G = \mathbb{Z}^m$ over the canonical ring $(\mathbb{Z}, +, \cdot)$ where G acts on a compact metric space. However, our state space X is not necessarily to be metrizable here. This point is important for our subsequent applications.

Theorem 1.2. *Given any discrete semimodule $(G, +)$ over a semiring $(R, +, \cdot)$, let $G \curvearrowright_\varphi X$ be a minimal t.d.s. over a compact Hausdorff space X . Then for any $T_1, \dots, T_l \in G$ and any open subset U of X , $U \neq \emptyset$, the multiple hitting-time set of U with itself under φ ,*

$$N_{T_1, \dots, T_l}(U) := \{t \in R: U \cap \varphi^{-tT_1}[U] \cap \dots \cap \varphi^{-tT_l}[U] \neq \emptyset\},$$

is syndetic in $(R, +)$ under the discrete topology.

Notes. 1. The statement still holds if instead G is a right semimodule by defining the hitting-time set as follows:

$$N_{T_1, \dots, T_l}(U) := \{t \in R: U \cap \varphi^{-T_1 t}[U] \cap \dots \cap \varphi^{-T_l t}[U] \neq \emptyset\},$$

for any $T_1, \dots, T_l \in G$.

2. In 1989 [8] Blaszczyk et al. proved that if G is a commutative semigroup of continuous transformations of a compact space X (i.e. G is a semimodule over $(\mathbb{Z}_+, +, \cdot)$) and it acts minimally on X , then for any non-empty open set U of X and any $T_1, \dots, T_l \in G$ there exists an integer $n \geq 1$ with $U \cap T_1^{-n}[U] \cap \dots \cap T_l^{-n}[U] \neq \emptyset$.
3. Moreover, the “syndeticity” of $N_{T_1, \dots, T_l}(U)$ did not appear in the original work of Furstenberg and Weiss in 1978 and nor in the work of Blaszczyk et al. in 1989 for \mathbb{Z}^m over $(\mathbb{Z}, +, \cdot)$ (cf. [17, Theorem 1.5], [18, Theorem 1.56], and [8, Theorem 1 and Corollary 2]).
4. Theorem 1.2 is never a consequence of Furstenberg’s Multiple Poincaré Recurrence Theorem ([16, 15]); this is because if $(R, +, \cdot)$ is uncountable, then we cannot reduce a Baire probability space $(X, \mathcal{B}a(X), \mu)$ to a standard Borel probability space by a factor map (to employ the disintegration of measures).

Here for any $t \in R$ and any open set $U \subset X$, we have that

$$U \cap \varphi^{-tT_1}[U] \cap \dots \cap \varphi^{-tT_l}[U] \neq \emptyset \Leftrightarrow \exists x_0 \in U \text{ s.t. } \varphi(tT_i, x_0) \in U, i = 1, \dots, l.$$

In this paper Theorem 1.1 is a consequence of Theorem 1.2. However, if we start with Theorem 1.1, then the following corollary follows at once from Corollary 3 of Theorem 1.1.

Corollary 1 (Multiple Recurrence in Open Covers). *Given any discrete semimodule $(G, +)$ over a discrete semiring $(R, +, \cdot)$, let $G \curvearrowright_\varphi X$ be a t.d.s. over a compact Hausdorff space X , and let \mathcal{U} be an open cover of X . Then there exists some $U \in \mathcal{U}$ such that for any $T_1, \dots, T_l \in G$, $N_{T_1, \dots, T_l}(U)$ is syndetic in $(R, +)$.*

Proof. Let X_0 be a minimal set of $G \curvearrowright_\varphi X$. Since there is some $U \in \mathcal{U}$ with $U \cap X_0 \neq \emptyset$ and $N_{T_1, \dots, T_l}(U \cap X_0) \subseteq N_{T_1, \dots, T_l}(U)$, hence by Theorem 1.2 for $G \curvearrowright_\varphi X_0$ it follows that $N_{T_1, \dots, T_l}(U)$ is syndetic in $(R, +)$. This proves the corollary. \square

This corollary in turn implies Theorem 1.2 by a standard homogeneity argument under the minimality hypothesis.

Next we will consider another application of Theorem 1.2 to pointwise multiple recurrence. For that we first need to recall and introduce some notions as follows:

Definition 1.3. Let $G \curvearrowright_\varphi X$ be a *t.d.s.* over a compact Hausdorff space X , where $(G, +)$ be a discrete semimodule over a discrete semiring $(R, +, \cdot)$. By \mathcal{U}_x it stands for the neighborhood system of X at x , for any given $x \in X$.

- (a) Given any $T \in G$, a point $p \in X$ is *uniformly recurrent* for $G \curvearrowright_\varphi X$ rel. T if

$$N_T(p, U) = \{t \in R \mid \varphi(tT, p) \in U\}, \quad \forall U \in \mathcal{U}_p,$$

is syndetic in $(R, +)$. See [19, 15].

- (b) A point $p \in X$ is called *multiply uniformly recurrent* for $G \curvearrowright_\varphi X$, if it is *multiply uniformly recurrent* for (T_1, \dots, T_l) , for any finite set $\{T_1, \dots, T_l\} \subseteq G$; i.e.,

$$N_{T_1, \dots, T_l}(p, U) = \{t \in R \mid \varphi(tT_i, p) \in U, i = 1, \dots, l\} \quad \forall U \in \mathcal{U}_p$$

is syndetic in $(R, +)$, for any $T_1, \dots, T_l \in G$.³

- (c) A point p is called *Bohr almost periodic* for $G \curvearrowright_\varphi X$ if the orbit closure $\text{cls}_X G_\varphi[p]$ is minimal for $G \curvearrowright_\varphi X$ and $\{\varphi(T, \cdot): X \rightarrow X\}_{T \in G}$ is an equicontinuous family restricted to $\text{cls}_X G_\varphi[p]$.
- (d) A point $p \in X$ is called a *multiply syndetic nonwandering point* for $G \curvearrowright_\varphi X$ if for any $T_1, \dots, T_l \in G$ and any $U \in \mathcal{U}_p$, $N_{T_1, \dots, T_l}(U)$ is syndetic in $(R, +)$.

Although there does not need to exist multiply uniformly recurrent points in general, yet we can easily obtain the following two statements from Theorem 1.2.

Corollary 2. *If $x \in X$ is a Bohr almost periodic point of $G \curvearrowright_\varphi X$, then it is multiply uniformly recurrent for $G \curvearrowright_\varphi X$.*

Proof. Write $Y = \text{cls}_X G_\varphi[p]$ and then the sub *t.d.s.* $G \curvearrowright_\varphi Y$ is minimal and equicontinuous. Let $U \in \mathcal{U}_p$ be arbitrarily given. Then by Theorem 1.2, it follows that for any $V \in \mathcal{U}_p$ and any $T_1, \dots, T_l \in G$, $N_{T_1, \dots, T_l}(V)$ is syndetic in $(R, +)$.

Since $\varphi: G \times Y \rightarrow Y$ is equicontinuous and Y is compact Hausdorff (a uniform space), we can take some $V \subset U$ so “small” that

$$\varphi(T, y) \in V \text{ for some } y \in V \text{ and some } T \in G \Rightarrow \varphi(T, x) \in U \quad \forall x \in V.$$

Now for any $t \in N_{T_1, \dots, T_l}(V)$, there is some point $y_t \in V$ with $\varphi(tT_i, y_t) \in V$ simultaneously for $i = 1, \dots, l$, and thus $\varphi(tT_i, p) \in U$ simultaneously for $i = 1, \dots, l$. This shows that $N_{T_1, \dots, T_l}(p, U)$ is syndetic in $(R, +)$ and thus p is multiply uniformly recurrent for $G \curvearrowright_\varphi X$.

The proof of Corollary 2 is therefore completed. \square

³A cyclic system (X, T) is called *multi-minimal* in [20, 10] if $(X^n, T \times T^2 \times \dots \times T^n)$ is minimal for all $n \geq 1$. In this case, every point $x \in X$ must be multiply uniformly recurrent for $\mathbb{Z}_+ \curvearrowright_T X$. However, it does not need to imply the multi-minimality of (X, T) that every point of X is multiply uniformly recurrent for $\mathbb{Z}_+ \curvearrowright_T X$.

Corollary 3. *Let $G \curvearrowright_\varphi X$ be a t.d.s. over a compact Hausdorff space X . Then there always exists a point $p \in X$ at which $G \curvearrowright_\varphi X$ is multiply syndetic nonwandering.*

Proof. This follows immediately from Theorem 1.2 and Zorn's lemma. \square

Note that if $(G, +)$ is a discrete group, then $(\mathbb{Z}, +)$ can be naturally imbedded in $(G, +)$ via the unit element 1 as a subgroup. However $\varphi: \mathbb{Z} \times X \rightarrow X$ does not need to be minimal. Although we may choose a minimal subset Λ of $\mathbb{Z} \curvearrowright_\varphi X$, yet we cannot obtain the multiple recurrence of the open complement $U = X \setminus \Lambda$ by the classical theorem of Furstenberg and Weiss [17, Theorem 1.5]. Moreover the arbitrariness of U in Theorem 1.2, corresponding to the arbitrariness of the choice of F , is important for proving Theorem 1.1 later. Because of this reason, our Theorem 1.2 is not a consequence of Furstenberg and Weiss [17, Theorem 1.5] generally.

In 1978 Furstenberg and Weiss proved their regional multiple recurrence theorem by using the multiple Birkhoff recurrence theorem and homogeneity. However, that idea is not workable in our situation; this is because there is no applicable pointwise multiple recurrence theorem for commuting maps on a non-metrizable compact Hausdorff space and moreover the multiple returning time set of a multiply recurrent point is not syndetic in general. Blaszczyk et al. in 1989 [8] gave a topological proof of the topological multidimensional van der Waerden theorem by using induction and the associated inverse system.

However, Theorem 1.2 will be proved in §2 under the guise of Theorem 2.2 following the nice idea of R. Ellis by using his enveloping semigroup theory. Then based on Theorem 1.2 together with a dynamics concept—weak central set—introduced later, we can show Theorem 1.1 in §3 using the idea of Bergelson, Furstenberg, Hindman and Katznelson 1989 [4]. We will end this paper with a closely related open question for our further study.

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2. The regional multiple recurrence theorem

In this section we will prove Theorem 1.2 following the framework of [4], also see [18, §1.11], completely different with [17, 8]. Let $(R, +, \cdot)$ be a discrete semiring and $(G, +)$ a discrete R -semimodule; and let

$$\varphi: G \times X \rightarrow X \quad \text{or simply write} \quad G \curvearrowright_\varphi X$$

be a t.d.s. over a compact Hausdorff space X in the ensuing arguments of this section. If we identify $(G, +)$ with the semigroup $(\{\varphi(g, \cdot)\}_{g \in G}, \circ)$ via

$$g \mapsto \varphi(g, \cdot) \quad \text{and} \quad g + h \mapsto \varphi^g \circ \varphi^h,$$

then X becomes a G -space.

2.1. A technical lemma

Given any integer $n \geq 1$, for any n elements T_1, T_2, \dots, T_n in G , by $\langle T_1, \dots, T_n \rangle_R$ we denote from here on the sub-semimodule of G , which is generated by T_1, \dots, T_n , over $(R, +, \cdot)$.

For any $g \in G$, define a continuous transformation of $X^n = \overbrace{X \times \cdots \times X}^{n\text{-times}}$ to itself

$$\hat{g}: X^n \rightarrow X^n; \quad (x_1, \dots, x_n) \mapsto (g(x_1), \dots, g(x_n)),$$

noting here that we have identified g with $\varphi(g, \bullet)$ based on the *t.d.s.* $G \curvearrowright_\varphi X$. On the other hand, define

$$\hat{T}^t = T_1^t \times \cdots \times T_n^t: X^n \rightarrow X^n; \quad (x_1, \dots, x_n) \mapsto (T_1^t(x_1), \dots, T_n^t(x_n)),$$

for any $t \in R$, where we have identified T_i^t with $\varphi(tT_i, \bullet)$ for $1 \leq i \leq n$ based on the *t.d.s.* $G \curvearrowright_\varphi X$.

It is a well-known fact that for any minimal subset Λ of $G \curvearrowright_\varphi X$, usually Λ does not need to be minimal for $\langle T_1, \dots, T_n \rangle_R \curvearrowright_\varphi X$; even nor for the classical case $R = \mathbb{Z}$ like Λ consists of a periodic orbit. Because of this reason the following is an important technical lemma for proving Theorem 1.2 later, our proof of which is mainly motivated by an argument due to R. Ellis for the case of $R = \mathbb{Z}$ (cf. [18, Proposition 1.55]).

Lemma 2.1. *Given any $T_1, \dots, T_n \in G$, let $\langle T_1, \dots, T_n \rangle_R \curvearrowright_\varphi X$ be minimal based on $G \curvearrowright_\varphi X$. Define two topological dynamical systems on X^n*

$$\begin{aligned} \xi: R \times X^n &\rightarrow X^n \text{ or write } R \curvearrowright_\xi X^n; & (t, \hat{x}) &\mapsto \hat{T}^t(\hat{x}) \\ \theta: \langle T_1, \dots, T_n \rangle_R \times X^n &\rightarrow X^n \text{ or write } \langle T_1, \dots, T_n \rangle_R \curvearrowright_\theta X^n; & (g, \hat{x}) &\mapsto \hat{g}(\hat{x}) \end{aligned}$$

and set

$$\mathfrak{T} = \{ \xi^t \theta^g = \hat{T}^t \hat{g}: X^n \rightarrow X^n \mid (t, g) \in R \times \langle T_1, \dots, T_n \rangle_R \}.$$

If Λ is a θ -minimal subset of X^n and set

$$\Sigma = \text{cls}_{X^n} \bigcup_{t \in R} \xi^t[\Lambda] = \text{cls}_{X^n} \xi(R \times \Lambda),$$

then Σ is a \mathfrak{T} -minimal subset of X^n .

Proof. Firstly, we can see that Σ is an \mathfrak{T} -invariant closed subset of X^n . Indeed, since for $1 \leq i \leq n$ we have $T_i^t g = \varphi(tT_i + g, \bullet) = gT_i^t: X \rightarrow X$ for all $g \in G, t \in R$ and so

$$(\hat{T}^t \hat{g})(\hat{T}^{t'} \hat{g}') = (\hat{T}^{t'} \hat{g}')(\hat{T}^t \hat{g}) \quad \forall (t, g), (t', g') \in R \times \langle T_1, \dots, T_n \rangle_R,$$

then \mathfrak{T} is an abelian multiplicative semigroup of continuous transformations of the n -fold product space X^n which is a compact Hausdorff space. Moreover,

$$\xi^t \theta^g (\xi^{t'} \theta^{g'}(z)) \in \xi^{t+t'} (\theta^{g+g'}[\Lambda]) = \xi^{t+t'}[\Lambda] \subseteq \Sigma$$

for any $(t, g), (t', g') \in R \times \langle T_1, \dots, T_n \rangle_R$ and $z \in \Lambda$.

Secondly, we let $\pi_i: (x_1, \dots, x_n) \mapsto x_i$ be the natural projection of Σ onto the i -th component X , for $i = 1, 2, \dots, n$. We now consider the action of the discrete semigroup \mathfrak{T} on the i -th component X via the representations

$$\theta^g \rightarrow g = \varphi(g, \bullet) \quad \text{and} \quad \xi^t \rightarrow T_i^t = \varphi(tT_i, \bullet) \quad \forall g \in \langle T_1, \dots, T_n \rangle_R \text{ and } t \in R;$$

that is, $\xi^t \theta^g(x_i) = \varphi(g + tT_i, x_i)$ for any $\xi^t \theta^g \in \mathfrak{T}$ and any $x_i \in X$. With respect to this action of \mathfrak{T} on X the map π_i is a \mathfrak{T} -homomorphism or \mathfrak{T} -factor map between (Σ, \mathfrak{T}) and (X, \mathfrak{T}) as follows:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\xi^t \theta^g} & \Sigma \\ \pi_i \downarrow & & \pi_i \downarrow \\ X & \xrightarrow{\xi^t \theta^g} & X \end{array} \quad \forall \xi^t \theta^g \in \mathfrak{T}.$$

Let $E(\Sigma, \mathfrak{T})$ be the Ellis enveloping semigroup of the subdynamical system (Σ, \mathfrak{T}) of (X^n, \mathfrak{T}) (cf., e.g., [13, 14, 1, 15, 18]). And we let $\pi_i^*: E(\Sigma, \mathfrak{T}) \rightarrow E(X, \mathfrak{T})$ be the corresponding homomorphism of the Ellis enveloping semigroups. Then notice that for this induced action of \mathfrak{T} on X , by

$$RT_i + \langle T_1, \dots, T_n \rangle_R = \langle T_1, \dots, T_n \rangle_R$$

it follows clearly that $E(X, \mathfrak{T}) = E(\langle T_1, \dots, T_n \rangle_R \curvearrowright_\varphi X)$ as closed subsets of the compact Hausdorff product space X^X , for all $i = 1, 2, \dots, n$.

Thirdly, we simply write $\hat{\mathbb{T}} = \{\theta^g(\cdot): X \rightarrow X\}_{g \in \langle T_1, \dots, T_n \rangle_R}$ that is an abelian multiplicative semigroup and let now u be any minimal idempotent of the Ellis enveloping semigroup $E(\Sigma, \hat{\mathbb{T}})$; that is, there exists a minimal closed left ideal J of $E(\Sigma, \hat{\mathbb{T}})$ such that u is an idempotent in J .

Choose v a minimal idempotent of $E(\Sigma, \mathfrak{T})$ in the closed left ideal $E(\Sigma, \mathfrak{T})u$ of $E(\Sigma, \mathfrak{T})$, i.e., there is a minimal closed left ideal I of $E(\Sigma, \mathfrak{T})$ such that v is an idempotent in I and $I \subseteq E(\Sigma, \mathfrak{T})u$; clearly $vu = v$, this is because $v = fu$ for some $f \in E(\Sigma, \mathfrak{T})$ and so $vu = fu^2 = fu = v$. Then by $vJ \subseteq J$, it follows that $v = vu \in J$. Next, we want to show that $uv = u$.

Indeed for each $i = 1, \dots, n$ set $u_i = \pi_i^*(u)$ and $v_i = \pi_i^*(v)$. As an element of $E(\Sigma, \mathfrak{T})$ is completely determined by its projections, to prove $uv = u$ it suffices to show that $u_i v_i = u_i$ for each $i = 1, \dots, n$.

Let $J_i = \pi_i^*(J)$ and $I_i = \pi_i^*(I)$ which both are minimal closed left ideals of the semigroup $E(\langle T_1, \dots, T_n \rangle_R \curvearrowright_\varphi X)$. Since for every i the map π_i^* is a semigroup homomorphism, hence $v_i u_i = v_i$, $I_i v_i = I_i$ and $J_i u_i = J_i$. In particular, by $v_i \in I_i \cap J_i$ and then $I_i = J_i$, we can see that v_i is an element of the minimal closed left ideal of $E(\langle T_1, \dots, T_n \rangle_R \curvearrowright_\varphi X)$ which contains u_i . This in turn implies that $u_i v_i = g_i v_i u_i = u_i$ for some $g_i \in I_i$ with $u_i = g_i v_i$ by an argument similar to that of $vu = v$ before and thus $uv = u$.

Therefore by $uv \in I$, it follows that u is an element of the minimal closed left ideal of $E(\Sigma, \mathfrak{T})$ which contains v , and then u is a minimal idempotent of $E(\Sigma, \mathfrak{T})$.

Finally, let x be an arbitrary point in Λ and then we can find some element u a minimal idempotent of $E(\Sigma, \hat{\mathbb{T}})$ with $u(\hat{x}) = \hat{x}$. From the above argument, it follows that u is also a minimal idempotent of $E(X, \mathfrak{T})$. Hence \hat{x} is a minimal point for the dynamical system (Σ, \mathfrak{T}) . Therefore, $\Sigma = \text{cls}_{X^n} \mathfrak{T}[\hat{x}]$ is a \mathfrak{T} -minimal subset of X^n .

The proof of Lemma 2.1 is therefore completed. \square

We notice that if G is a right R -semimodule, then we need to define ξ in Lemma 2.1 via $T_i^t = \varphi(T_i t, \cdot)$.

2.2. Topological multidimensional van der Waerden Theorem

With Lemma 2.1 at hands, now we can readily prove our regional multiple recurrence theorem for any semimodules acting on a compact Hausdorff space X .

Theorem 1.2 is obviously equivalent to the following. This kind of result is also called Topological Multidimensional van der Waerden Theorem in the literature; see, e.g., [3, 8].

Theorem 2.2. *Given any discrete semimodule $(G, +)$ over a semiring $(R, +, \cdot)$, let $G \curvearrowright_\varphi X$ be a t.d.s. over the compact Hausdorff space X . If $G \curvearrowright_\varphi X$ is minimal, then for any $n \geq 1$, any $T_1, \dots, T_n \in G$ and any non-empty open subset U of X , the multiple hitting-time set of U with itself by the action of φ ,*

$$N_{T_1, \dots, T_n}(U) = \{t \in R: \varphi^{-tT_1}[U] \cap \dots \cap \varphi^{-tT_n}[U] \neq \emptyset\},$$

is syndetic in $(R, +)$ under the discrete topology.

Proof. Let $G \curvearrowright_\varphi X$ be a minimal *t.d.s.* over the compact Hausdorff space X as in Theorem 2.2 and let $T_1, \dots, T_n \in G$ be any given. Let U be an arbitrary open subset of X with $U \neq \emptyset$.

By Zorn's lemma, it follows that we can choose some minimal subset X_0 of X for the *t.d.s.* $\langle T_1, \dots, T_n \rangle_R \curvearrowright_\varphi X$ with $U \cap X_0 \neq \emptyset$; noting that $\langle T_1, \dots, T_n \rangle_R$ is a sub-semimodule of $(G, +)$ over $(R, +, \cdot)$. Indeed, let $Y_0 \subset X$ be any minimal set of $\langle T_1, \dots, T_n \rangle_R \curvearrowright_\varphi X$; then by the minimality of $G \curvearrowright_\varphi X$, it follows that for any given $y_0 \in Y_0$ there is some element $g_0 \in G$ such that $x_0 := \varphi(g_0, y_0)$ belongs to U . Whence by the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\varphi(g, \cdot)} & X \\ \downarrow \pi & & \downarrow \pi \\ X & \xrightarrow{\varphi(g, \cdot)} & X \end{array} \quad \forall g \in \langle T_1, \dots, T_n \rangle_R, \quad \text{where } \pi: x \mapsto \varphi(g_0, x),$$

we can see that the minimal set of $\langle T_1, \dots, T_n \rangle_R \curvearrowright_\varphi X$

$$X_0 := \text{cls}_X \{ \varphi(g, x_0) : g \in \langle T_1, \dots, T_n \rangle_R \}$$

satisfies the desired property.

Now, without loss of generality, we may assume that $\langle T_1, \dots, T_n \rangle_R \curvearrowright_\varphi X$ is minimal. Let ξ, θ and \mathfrak{T} all be defined as in Lemma 2.1. Now set

$$\Lambda = \Delta_n(X) = \{ (x, x, \dots, x) \in X^n \}$$

and

$$\Sigma = \text{cls}_{X^n} \bigcup_{t \in R} \xi^t [\Delta_n(X)].$$

Since $\langle T_1, \dots, T_n \rangle_R \curvearrowright_\varphi X$ is minimal by hypothesis, then $\Delta_n(X)$ is θ -minimal in X^n . By Lemma 2.1, the point

$$\hat{x} = (x, x, \dots, x) \in \Delta_n(X) \quad \text{with } x \in U$$

is a minimal point (or called a uniformly recurrent point and a von Neumann almost periodic point in [15] and [19], respectively) of the topological dynamical system $R \times \langle T_1, \dots, T_n \rangle_R \curvearrowright_{\mathfrak{T}} \Sigma$. Hence the returning-time set

$$N_{\mathfrak{T}}(\hat{x}, \widehat{U}) = \left\{ (t, g) \in R \times \langle T_1, \dots, T_n \rangle_R : \xi^t \theta^g(\hat{x}) \in \widehat{U} \right\},$$

where $\widehat{U} = U \times \dots \times U$ and where $R \times \langle T_1, \dots, T_n \rangle_R$ is thought of as a discrete additive semigroup, is a “syndetic” subset of $(R \times \langle T_1, \dots, T_n \rangle_R, +)$ in the sense that one can find a finite subset $K = \{(s_1, g_1), \dots, (s_k, g_k)\}$ of $R \times \langle T_1, \dots, T_n \rangle_R$ so that

$$(K + \tau) \cap N_{\mathfrak{T}}(\hat{x}, \widehat{U}) \neq \emptyset \quad \forall \tau \in R \times \langle T_1, \dots, T_n \rangle_R;$$

see, e.g., [9]. Thus for any $(t, g) \in N_{\mathfrak{T}}(\hat{x}, \widehat{U})$ we have

$$\varphi(tT_i, \varphi(g, x)) \in U, \quad i = 1, \dots, n.$$

Therefore,

$$\varphi(g, x) \in \bigcap_{i=1}^n \varphi^{-iT_i}[U] \neq \emptyset, \quad \forall (t, g) \in N_{\mathcal{T}}(\hat{x}, \hat{U}).$$

Let

$$S = \text{Pr}_1 \left[N_{\mathcal{T}}(\hat{x}, \hat{U}) \right], \quad \text{where } \text{Pr}_1 : (t, g) \mapsto t.$$

Then $S \subseteq N_{T_1, \dots, T_n}(U)$ and S is syndetic in $(R, +)$.

This thus completes the proof of Theorem 2.2. \square

Theorem 2.2 will not only imply Theorem 1.2 but also the classical multiple Birkhoff recurrence theorem of Furstenberg and Weiss ([15, Theorem 2.6]), if X is a metric space.

Question 2.3. *If $(R, +, \cdot) = (\mathbb{Z}_+, +, \cdot)$, then $N_{T_1, \dots, T_l}(U)$ in Theorem 2.2 should be an IP^* -set; in other words, $N_{T_1, \dots, T_l}(U)$ should intersect every IP -set in $(\mathbb{Z}_+, +)$.*

We note that if X is a compact metric space, then by [15, Theorem 2.16] and homogeneity we can see that $N_{T_1, \dots, T_l}(U)$ is an IP^* -set in \mathbb{Z}_+ . However, in the present non-metrizable situation, there does not need to exist a multiply recurrent point for (T_1, \dots, T_l) (cf. [2] for counterexamples).

3. Weak central sets and van der Waerden theorem

This section will be mainly devoted to proving the van der Waerden theorem (Theorem 1.1) based on the Regional Multiple Recurrence Theorem (Theorem 1.2) and Weak Central Sets of discrete semigroups introduced below. Moreover we will consider van der Waerden subset of any semimodule.

3.1. Uniform recurrence of motions

Let $(G, +)$ be a discrete abelian semigroup unless an explicit declaration and moreover let $\varphi: G \times X \rightarrow X$, or write $G \curvearrowright_{\varphi} X$, be a topological dynamical system (*t.d.s.*) over a compact Hausdorff space X .

Recall that a point $y \in X$ is called *uniformly recurrent* for $G \curvearrowright_{\varphi} X$ (cf., e.g., [15, 9]), provided that for any neighborhood U of y the “return-times” set

$$N_{\varphi}(y, U) = \{g \in G \mid \varphi(g, y) \in U\}$$

is syndetic in $(G, +)$; that is to say, there is a finite subset K of G such that

$$N_{\varphi}(y, U) \cap (g + K) \neq \emptyset$$

for all $g \in G$. Since here we have endowed G with the discrete topology, this kind of syndeticity is the strongest.

Under our assumption that X is a compact Hausdorff space, it is easy to check that $y \in X$ is uniformly recurrent for $G \curvearrowright_{\varphi} X$ if and only if the orbit closure $\text{cls}_X G_{\varphi}[y]$ is minimal in X with $y \in \text{cls}_X G_{\varphi}[y]$ (cf., e.g., [9]).

Let (\mathbf{K}, \diamond) be a compact Hausdorff group and $\psi: G \times X \rightarrow \mathbf{K}$ a continuous map. We shall say that ψ has the *cocycle property based on $G \curvearrowright_\varphi X$* provided that

$$\psi(s + t, x) = \psi(t, \varphi(s, x)) \diamond \psi(s, x) \quad \forall s, t \in G \text{ and } x \in X.$$

We can then define a *t.d.s.* on the product topological space $X \times \mathbf{K}$:

$$\Phi: G \times X \times \mathbf{K} \rightarrow X \times \mathbf{K} \text{ or } G \curvearrowright_\Phi X \times \mathbf{K}; \quad (t, (x, k)) \mapsto (\varphi(t, x), \psi(t, x) \diamond k).$$

The resulting system $G \curvearrowright_\Phi X \times \mathbf{K}$ is called a *group extension* of $G \curvearrowright_\varphi X$ via ψ .

The following result is due to Hillel Furstenberg for the very important case $G = \mathbb{N}$ (cf. [15, Theorem 1.19]).

Theorem 3.1. *Let $G \curvearrowright_\Phi X \times \mathbf{K}$ be the group extension of $G \curvearrowright_\varphi X$ via a cocycle $\psi: G \times X \rightarrow \mathbf{K}$. If $x_0 \in X$ is a uniformly recurrent point of $G \curvearrowright_\varphi X$, then for each $k_0 \in \mathbf{K}$, (x_0, k_0) is uniformly recurrent for $G \curvearrowright_\Phi X \times \mathbf{K}$.*

Proof. Since $\text{cls}_X G_\varphi[x_0]$ is minimal for $G \curvearrowright_\varphi X$ in X , we may assume that, to begin with, $G \curvearrowright_\varphi X$ is minimal itself. Now by Zorn's lemma, let $Z \subseteq X \times \mathbf{K}$ be minimal for $G \curvearrowright_\Phi X \times \mathbf{K}$. Since $\pi: (x, k) \mapsto x$ from $G \curvearrowright_\Phi X \times \mathbf{K}$ onto $G \curvearrowright_\varphi X$ is a homomorphism, then $\pi(Z) = X$ and so $\{x_0\} \times \mathbf{K} \cap Z \neq \emptyset$. Thus there exists some (x_0, k_0) in $\{x_0\} \times \mathbf{K}$ which is uniformly recurrent for $G \curvearrowright_\Phi X \times \mathbf{K}$. On the other hand, for any $k' \in \mathbf{K}$, $R_{k'}: X \times \mathbf{K} \rightarrow X \times \mathbf{K}$ by $(x, k) \mapsto (x, k \diamond k')$ is an automorphism of $G \curvearrowright_\Phi X \times \mathbf{K}$; this then implies that each $(x_0, k) \in \{x_0\} \times \mathbf{K}$ is uniformly recurrent for $G \curvearrowright_\Phi X \times \mathbf{K}$. \square

Corollary 1. *Let $\psi: G \times X \rightarrow \mathbf{K}$ be a cocycle based on $\varphi: G \times X \rightarrow X$. If $x_0 \in X$ is a uniformly recurrent point of $G \curvearrowright_\varphi X$, then the hitting-time set*

$$N_\psi(x_0, V_e) = \{t \in G: \psi(t, x_0) \in V_e\}$$

is syndetic in $(G, +)$, for any neighborhood V_e of the identity e of (\mathbf{K}, \diamond) .

A question naturally arising in dynamical systems is about the inheritance: does a G -action $G \curvearrowright_\varphi X$ have the same recurrence as each g -sample maps $\varphi(g, \cdot)$? It is a well-known fact that if a point x_0 is recurrent for a \mathbb{C}^0 -flow $\varphi: \mathbb{R} \times X \rightarrow X$, so is it for the 1-sample map $\varphi(1, \cdot): X \rightarrow X$. An analogue holds for uniform recurrence.

Corollary 2. *If $x_0 \in X$ is a uniformly recurrent point of a \mathbb{C}^0 -semi flow $\varphi: \mathbb{R}_+ \times X \rightarrow X$ where $\mathbb{R}_+ := [0, +\infty)$ is topologized by the Euclidean metric, then for any $\tau > 0$, x_0 is also a uniformly recurrent point for the τ -sample map $\varphi(\tau, \cdot): X \rightarrow X$.*

Proof. Given any $\tau > 0$, let $\mathbb{T}_\tau = \mathbb{R}/\tau\mathbb{Z}$, the additive group of reals modulo τ . Define the cocycle driven by φ as follows:

$$\psi: \mathbb{R}_+ \times X \rightarrow \mathbb{T}_\tau \quad \text{by } (t, x) \mapsto \psi(t, x) = t \pmod{\tau}.$$

Clearly x_0 is also uniformly recurrent for $\mathbb{R}_+ \curvearrowright_\varphi X$ under the discrete topology of \mathbb{R}_+ (cf. [15, 9]). Then by Theorem 3.1, it follows that $(x_0, 0)$ is uniformly recurrent for the group extension $\mathbb{R}_+ \curvearrowright_\Phi X \times \mathbb{T}_\tau$ of φ via ψ .

Let U be an arbitrary open neighborhood of x_0 in X . Since $\varphi(t, x)$ is jointly continuous with respect to $t \in \mathbb{R}_+$ (with the Euclidean metric), $x \in X$, and X is compact Hausdorff, one can find a compact neighborhood V of x_0 with $V \subset U$ and an $\varepsilon > 0$ with $\varepsilon \ll 1$ such that

$$\varphi(t, x) \in U \quad \forall x \in V \text{ and } 0 \leq t \leq \varepsilon.$$

Next by Theorem 3.1, the set

$$N_\Phi(x_0, V \times [0, \varepsilon)) = \{t \in \mathbb{R}_+ \mid \varphi(t, x_0) \in V, t \in [0, \varepsilon) \bmod \tau\}$$

is syndetic in $(\mathbb{R}_+, +)$ under the discrete topology. Since for any $t \in N_\Phi(x_0, V \times [0, \varepsilon))$ we may write t as

$$t = n_i \tau - r \quad \text{where } n_i \in \mathbb{N}, 0 \leq r < \varepsilon,$$

hence

$$\varphi(n_i \tau, x_0) = \varphi(r, \varphi(t, x_0)) \in U$$

which implies that

$$N_{\varphi(\tau, \cdot)}(x_0, U) = \{n \in \mathbb{N} \mid \varphi(n\tau, x_0) \in U\}$$

is syndetic in $(\mathbb{Z}_+, +)$. This completes the proof of Corollary 2. \square

It should be noted that the joint continuity of $\varphi(t, x)$ under the Euclidean metric of \mathbb{R}_+ plays a role in the above proof. Moreover the multi-dimensional version of Corollary 2 also holds by a similar argument:

Corollary 3. *If $x_0 \in X$ is a uniformly recurrent point of a C^0 -semi flow $\varphi: \mathbb{R}_+^d \times X \rightarrow X$ where \mathbb{R}_+^d is topologized by the Euclidean metric, then x_0 is also a uniformly recurrent point for $\varphi: \mathbb{Z}_+^d \times X \rightarrow X$.*

3.2. Group extensions of uniquely ergodic systems

Let X be a compact metric space and \mathbf{K} a compact second countable Hausdorff group in the sequel of this subsection. Let $m_{\mathbf{K}}$ be the Haar measure of \mathbf{K} with $m_{\mathbf{K}}(\mathbf{K}) = 1$. Then for any invariant Borel probability measure ν of a t.d.s. $G \curvearrowright_\varphi X$ on X , it is easy to see that $\nu \otimes m_{\mathbf{K}}$ is invariant for any group extension $G \curvearrowright_\Phi X \times \mathbf{K}$ via a cocycle $\psi: G \times X \rightarrow \mathbf{K}$ (cf., e.g., [15, §3.3]). We now consider the following natural question.

Question 3.2. *Let $G \curvearrowright_\varphi (X, \nu)$ be uniquely ergodic and let $G \curvearrowright_\Phi (X \times \mathbf{K}, \nu \otimes m_{\mathbf{K}})$ be ergodic. Is $G \curvearrowright_\Phi (X \times \mathbf{K}, \nu \otimes m_{\mathbf{K}})$ uniquely ergodic?*

Notice that when $G = \mathbb{Z}$ or \mathbb{N} , this question is confirmative by [15, Proposition 3.10] via classical ergodic theory of cyclic topological dynamical systems. By an almost same framework of the proof of [15, Proposition 3.10] (also [12, Theorem 4.21]) in view of [11], we can obtain the following generalization of Furstenberg's theorem:

Theorem 3.3. *Given any amenable group G with a left Haar measure m_G , let $G \curvearrowright_\varphi X$ be uniquely ergodic with an invariant Borel probability measure ν , and let $G \curvearrowright_\Phi X \times \mathbf{K}$ be the group extension of $G \curvearrowright_\varphi X$ via a cocycle $\psi: G \times X \rightarrow \mathbf{K}$ such that $G \curvearrowright_\Phi (X \times \mathbf{K}, \nu \otimes m_{\mathbf{K}})$ is ergodic. Then $G \curvearrowright_\Phi (X \times \mathbf{K}, \nu \otimes m_{\mathbf{K}})$ is uniquely ergodic.*

Proof. Let $\{F_\theta; \theta \in \Theta\}$ be any given L^∞ -admissible Følner net of G (cf. [11, Theorem 0.6]). We now consider the ergodic action $G \curvearrowright_\Phi (X \times \mathbf{K}, \nu \otimes m_{\mathbf{K}})$. Since $\{F_\theta; \theta \in \Theta\}$ is L^∞ -admissible for G , hence for every point $(x_0, k_0) \in X \times \mathbf{K}$ we have, in the sense of Moore-Smith net limit,

$$\lim_{\theta \in \Theta} \frac{1}{|F_\theta|} \int_{F_\theta} f(\Phi(g, (x_0, k_0))) dg = f^*(x_0, k_0) \quad \forall f \in C(X \times \mathbf{K}).$$

On the other hand, by [11, Theorem 0.6], we can obtain that, for any $f \in C(X \times \mathbf{K})$, it holds that

$$f^*(x_0, k_0) = \int_{X \times \mathbf{K}} f d\nu \otimes m_{\mathbf{K}} \quad (\nu \otimes m_{\mathbf{K}}\text{-a.e. } (x_0, k_0) \in X \times \mathbf{K}).$$

Because $X \times \mathbf{K}$ is a compact metric space, $C(X \times \mathbf{K})$ is separable under the uniform topology and then for $\nu \otimes m_{\mathbf{K}}$ -a.e. $(x_0, k_0) \in X \times \mathbf{K}$ we have

$$f^*(x_0, k_0) = \int_{X \times \mathbf{K}} f d\nu \otimes m_{\mathbf{K}} \quad \forall f \in C(X \times \mathbf{K}).$$

That is to say, almost every (x_0, k_0) is generic for $G \curvearrowright_\Phi (X \times \mathbf{K}, \nu \otimes m_{\mathbf{K}})$.

Since $R_{k'} : (x, k) \mapsto (x, kk')$, for each $k' \in \mathbf{K}$, is an automorphism of $G \curvearrowright_\Phi (X \times \mathbf{K}, \nu \otimes m_{\mathbf{K}})$, we can see that for each $f \in C(X \times \mathbf{K})$,

$$f^*(x_0, k) = \int_{X \times \mathbf{K}} f d\nu \otimes m_{\mathbf{K}} \quad \forall k \in \mathbf{K}$$

for ν -a.e. $x_0 \in X$.

Similarly we can show that if μ is any Φ -ergodic Borel probability measure on the compact metric space $X \times \mathbf{K}$, then for any $f \in C(X \times \mathbf{K})$

$$f^*(x_0, k_0) = \int_{X \times \mathbf{K}} f d\mu \quad (\mu\text{-a.e. } (x_0, k_0) \in X \times \mathbf{K}).$$

Therefore by the fact that μ and $\nu \otimes m_{\mathbf{K}}$ both cover ν , it follows that

$$\int_{X \times \mathbf{K}} f d\mu = \int_{X \times \mathbf{K}} f d\nu \otimes m_{\mathbf{K}} \quad \forall f \in C(X \times \mathbf{K}).$$

Hence $\mu = \nu \otimes m_{\mathbf{K}}$. This concludes the proof of Theorem 3.3. \square

Note that from the above proof we can see that if $G \curvearrowright_\Phi X$ is not uniquely ergodic, then $G \curvearrowright_\Phi X \times \mathbf{K}$ is not uniquely ergodic too; but $\nu \otimes m_{\mathbf{K}}$ is just the unique ergodic measure that covers ν by the projection $\pi : (x, k) \mapsto x$. In the context of Theorem 3.3, every point $x \in X$ is generic for $G \curvearrowright_\Phi (X, \nu)$ relative to any L^∞ -admissible Følner net for G .

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the additive group of reals modulo 1. Motivated by Weyl [26] or [15, §3.3] we can easily get the following

Proposition 3.4. *Let $p(x) = a_d x^d + \cdots + a_1 x + a_0$ be a real polynomial with $a_d \neq 0$. Then the sequence $\langle p(x) \rangle_{x \geq 0}$ modulo one is equi-distributed on \mathbb{T} in the sense that*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f(p(x)) dx = \int_0^1 f(\theta) d\theta$$

for any continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$.

Proof. Let $p_0(x) = a_1x + \dots + a_dx^d \in \mathbb{R}[x]$. Since $p_0(\mathbb{R}_+) = \mathbb{T} \bmod 1$, then from [24] it follows that $\langle p_0(x) \rangle_{x \geq 0} \bmod 1$ is equi-distributed on $[0, 1)$. Now for any $f \in C(\mathbb{T})$, let $g(\theta) = f(a_0 + \theta)$. Then $g \in C(\mathbb{T})$ and so

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f(p(x)) dx = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T g(p_0(x)) dx = \int_0^1 f(a_0 + \theta) d\theta = \int_0^1 f(\theta) d\theta.$$

This concludes the proof of Proposition 3.4. \square

3.3. \mathbb{T} -extension of Bohr almost periodic points

Let us return to a classical cyclic dynamical system (X, T) where T is a continuous transformation of a compact metric space X . Recall that a point $x_0 \in X$ is called a *Bohr almost periodic point* for T (in one of its equivalent formulations) if x_0 is uniformly recurrent for T and T restricted to the orbit-closure $\text{cls}_X\{T^n x_0 : n \geq 0\}$ is equicontinuous (cf. [22, Definition V8.01, Theorem V8.05]).

It is well known that the group extension of a recurrent (resp. uniformly) point of (X, T) are recurrent (resp. uniformly) points (cf. [15, Theorems 1.4 and 1.19]). However, there is no analogous arguments for the group extension of Bohr almost periodic points in the literature. Here we are going to consider the \mathbb{T} -extension of Bohr almost periodic points.

Theorem 3.5. *Let T be a continuous transformation of a compact metric space X , $x_0 \in X$ a Bohr almost periodic point for T with $X = \text{cls}_X\{T^n x_0 : n > 0\}$, and $\psi : X \rightarrow \mathbb{T}$ a continuous function. Then the group extension*

$$T_\psi : X \times \mathbb{T} \rightarrow X \times \mathbb{T}; \quad (x, \theta) \mapsto (Tx, \theta + \psi(x))$$

is either minimal itself or such that each $(x, \theta) \in X \times \mathbb{T}$ is Bohr almost periodic for T_ψ and the cohomology equation $h\psi = \xi \circ T - \xi$ has a solution (h, ξ) in $\mathbb{N} \times C^0(X, \mathbb{T})$.

Proof. If $(X \times \mathbb{T}, T_\psi)$ is minimal itself, then we can stop here; next we assume that $X \times \mathbb{T}$ is not minimal for T_ψ . Let Z be a minimal T_ψ -invariant closed subset of $X \times \mathbb{T}$ by Zorn's lemma. Then $Z \subsetneq X \times \mathbb{T}$.

Since the projection of Z to X , via $\pi : (x, \theta) \mapsto x$, is T -invariant and compact and by hypothesis (X, T) is minimal, we must have $\pi(Z) = X$. Consequently, the set $Z_x = \{\theta \in \mathbb{T} : (x, \theta) \in Z\}$ is nonempty for each $x \in X$. We shall show that there exists a number $h \in \mathbb{N}$ such that Z_x consists of a coset of the group \mathbb{T} modulo the subgroup $\{0, 1/h, 2/h, \dots, (h-1)/h\}$. Namely, let

$$H = \{\phi : R_\phi[Z] \cap Z \neq \emptyset\}$$

where $R_\phi : X \times \mathbb{T} \rightarrow X \times \mathbb{T}$ is defined by $(x, \theta) \mapsto (x, \theta + \phi)$ for each $\phi \in \mathbb{T}$. Since R_ϕ commutes with T_ψ , $R_\phi[Z]$ is again T_ψ -minimal and so if $R_\phi[Z] \cap Z \neq \emptyset$, we must have $R_\phi[Z] = Z$. It follows that H is a closed subgroup of \mathbb{T} . Moreover, unless $Z = X \times \mathbb{T}$, $H \neq \mathbb{T}$. Hence by Haar's theorem, it follows that $H = \{0, 1/h, \dots, (h-1)/h\}$ for some $h \in \mathbb{N}$. In view of the two characterization of H :

$$H = \{\phi : R_\phi[Z] \cap Z \neq \emptyset\} = \{\phi : R_\phi[Z] = Z\},$$

we can see that $Z_x = \theta_x + H$ for some $\theta_x \in \mathbb{T}$ and that H is independent of the choice of $x \in X$.

Let $h\theta = \overbrace{\theta + \dots + \theta}^{h\text{-times}}$ for any $\theta \in \mathbb{T}$. Now define $S_h: (x, \theta) \mapsto (x, h\theta)$ from $X \times \mathbb{T}$ onto itself and define a dynamical system

$$T_{h\psi}: X \times \mathbb{T} \rightarrow X \times \mathbb{T} \quad \text{by } (x, \theta) \mapsto (Tx, \theta + h\psi(x)).$$

Clearly $T_{h\psi}S_h = S_hT_\psi$ so that S_h is a homomorphism of $(X \times \mathbb{T}, T_\psi)$ onto $(X \times \mathbb{T}, T_{h\psi})$. It follows that $S_h[Z]$ is a $T_{h\psi}$ -invariant subset of $X \times \mathbb{T}$. Moreover, all the fibers $S_h[Z]_x$ consist of singletons, since $S_h[Z]_x = hZ_x = h\theta_x$ modulo 1. This means that $S_h[Z]$ is the graph of a continuous function $\xi: x \mapsto \theta_x$ of X to \mathbb{T} . To say that $S_h[Z]$ is $T_{h\psi}$ -invariant is to say that

$$\xi(Tx) = \xi(x) + h\psi(x) \quad \forall x \in X.$$

Replacing ξ by $h^{-1}\xi$, we can see that there exists some step function $j: X \rightarrow \{0, 1, \dots, h-1\}$ such that

$$\psi(x) = \xi(Tx) - \xi(x) + \frac{j(x)}{h} \pmod{1} \quad \forall x \in X.$$

Since ψ, ξ both are continuous, hence $j(x) \equiv j_0$ for some integer j_0 with $0 \leq j_0 \leq h-1$. Whence

$$\psi(x) = \xi(Tx) - \xi(x) + \frac{j_0}{h} \pmod{1} \quad \forall x \in X.$$

This implies that $\{T_\psi^n: X \times \mathbb{T} \rightarrow X \times \mathbb{T}\}_{n \in \mathbb{N}}$ is equicontinuous.

Indeed, for any $n > 0$, $T_\psi^n(x, \theta) = (T^n x, \theta + \psi(n, x))$, where the cocycle $\psi(n, x)$ is such that

$$\psi(n, x) = \psi(x) + \psi(Tx) + \dots + \psi(T^{n-1}x)$$

and then

$$\psi(n, x) = \xi(T^n x) - \xi(x) + \frac{j_n}{h} \pmod{1},$$

where $j_n \in \{0, 1, \dots, h-1\}$ depends only upon n . Since $\{T^n: X \rightarrow X\}_{n \in \mathbb{N}}$ is equicontinuous by the hypothesis, then $\{T_\psi^n\}_{n \in \mathbb{N}}$ is equicontinuous from $X \times \mathbb{T}$ to itself.

Finally by Theorem 3.1, each $(x, \theta) \in X \times \mathbb{T}$ is uniformly recurrent and thus Bohr almost periodic for T_ψ . This proves Theorem 3.5. \square

The above proof of Theorem 3.5 is inspired by that of [15, Lemma 1.25].

3.4. Weak central sets of semigroups

The following concept is a slight generalization of Furstenberg central sets of \mathbb{N} ([15, Definition 8.3]).

Definition 3.6. A subset S of a discrete semigroup $(G, +)$ is referred to as a *weak central set* of $(G, +)$, where G is not necessarily commutative, if

- there exists a *t.d.s.* $\varphi: G \times X \rightarrow X$ over a compact Hausdorff space X ,
- a point $x \in X$ and a uniformly recurrent point $y \in X$ of $G \curvearrowright_\varphi X$ that is *weakly proximal* to x under φ in the sense that $\text{cls}_X G_\varphi[x] \cap \text{cls}_X G_\varphi[y] \neq \emptyset$,

- and there is an open neighborhood U of y ,

such that $S = \{g \in G: \varphi(g, x) \in U\}$.

It should be noticed here that $\{x, y\}$ is not necessarily to be a classical proximal pair for $G \curvearrowright_\varphi X$ in our Def. 3.6.

Clearly, G is itself a weak central set of $(G, +)$ by considering a singleton dynamical system. If x is itself uniformly recurrent for $G \curvearrowright_\varphi X$, then $N_\varphi(x, U)$ is a syndetic weak central set of $(G, +)$. Of course, a weak central set need not be syndetic in general.

Next we will present some basic combinatorial properties of weak central sets, which are generalizations of properties of Furstenberg central sets (cf. e.g., [15, Proposition 8.9]).

Lemma 3.7. *Let S be a weak central set of a discrete semigroup $(G, +)$ not necessarily commutative. Then it is “piecewise syndetic” in $(G, +)$; that is, one can find a syndetic subset S' of G such that for any finite subset A of S' , there is some $g_a \in S$ with $A + g_a \subset S$. So if G is a group, then $S - S$ contains a syndetic subset of G .*

Proof. This follows at once from Def. 3.6. □

It should be noted here that the associated $t.d.s.$ $\varphi: G \times X \rightarrow X$ in Def. 3.6 is such that

$$\varphi(g + h, x) = \varphi(g, \varphi(h, x)) \quad \forall g, h \in G, x \in X.$$

However, if we previously assume that

$$\varphi(g + h, x) = \varphi(h, \varphi(g, x)) \quad \forall g, h \in G, x \in X,$$

then we can only obtain $g_a + A \subset S$ in Lemma 3.7.

Lemma 3.8. *Let S be a weak central set of a discrete semimodule $(G, +)$ over a discrete semiring $(R, +, \cdot)$. Then for any finite set $F \subseteq G$ one can find a syndetic subset N_F of $(R, +)$, which contains an IP-set, such that for each $d \in N_F$ there exists an associated element $a \in S$ with $a + dF \subseteq S$.*

Proof. Let $\varphi: G \times X \rightarrow X$ or write $G \curvearrowright_\varphi X$ be a $t.d.s.$ over a compact Hausdorff space X , $y \in X$ a uniformly recurrent point of $G \curvearrowright_\varphi X$ weakly proximal to some point $x \in X$, U an open neighborhood of y in X such that $S = \{g \in G | \varphi(g, x) \in U\}$.

Given any finite subset F of G , write $F = \{T_1, \dots, T_l\}$. Since y is a uniformly recurrent point of $G \curvearrowright_\varphi X$, the orbit closure $cls_X G_\varphi[y]$ of y is φ -minimal in X . For simplicity, set $Y = cls_X G_\varphi[y]$.

Then by applying Theorem 1.2 with the minimal $t.d.s.$ $\varphi: G \times Y \rightarrow Y$, it follows that there exists some syndetic subset N_F of $(R, +)$, which contains an IP-set, such that for any $d \in N_F$, one can find a point $y' = y'(b) \in U \cap Y$ with $\varphi(dT_i, y') \in U$ simultaneously for $i = 1, 2, \dots, l$. In addition, since x is weakly proximal to y under φ and Y is minimal for $G \curvearrowright_\varphi X$, hence $Y \subseteq cls_X G_\varphi[x]$ and then one can find some $a \in S$ such that $\varphi(a, x) \in U$ is so close to y' that $\varphi(dT_i + a, x) \in U$ for $1 \leq i \leq l$.

This thus completes the proof of Lemma 3.8. □

According to the Note of Theorem 1.2, if we consider $a + Fd$ instead of $a + dF$, then the statement of Lemma 3.8 still holds for any right semimodules.

Comparing with Furstenberg’s topological discussion of the case $G = \mathbb{Z}$ or \mathbb{N} (cf. [15, Proposition 8.9]), in the proof of Lemma 3.8 we have overcome the following two obstructions by using weak central set of a semiring and Theorem 1.2 proved before:

- Since our underlying space X is not necessarily a (compact) metric space, we cannot find a Lebesgue number ε and there is no the classical proximality here.
- For our situation here, there exists no an applicable pointwise Multiple Birkhoff Recurrence Theorem (cf. [17], also [15, Theorem 2.6]).

As a central set in \mathbb{N} must be a weak central set, our Lemma 3.8 is a generalization of Furstenberg [15, Proposition 8.9].

The following lemma is a standard result by Furstenberg's correspondence principle, which is also valid for any discrete semigroup with a zero element \mathbf{o} .

Lemma 3.9. *Let $(G, +)$ be a discrete semimodule over a semiring $(R, +, \cdot)$. Then in a finite partition $G = B_1 \cup \dots \cup B_q$ with $B_j \neq \emptyset$ for each $j = 1, \dots, q$, one of the sets B_j is a weak central set of $(G, +)$.*

Proof. Over the standard compact Hausdorff space $X := \{1, \dots, q\}^G$ where $\{1, \dots, q\}$ is discrete and each $x \in X$ is thought of as a function $x(\cdot): G \rightarrow \{1, \dots, q\}$, we form the topological dynamical system

$$\varphi: G \times X \rightarrow X \quad \text{or} \quad G \curvearrowright_\varphi X$$

in the following ways:⁴

$$(g, x(\cdot)) \mapsto \varphi(g, x(\cdot)) = x(\cdot + g), \quad \forall x(\cdot): G \rightarrow \{1, \dots, q\} \text{ and } g \in G;$$

here $x(\cdot + g): G \rightarrow \{1, \dots, q\}$ is given by $t \mapsto x(t + g)$ for any $t \in G$. Let $\xi(\cdot): G \rightarrow \{1, \dots, q\}$ be defined by

$$\xi(g) = i \Leftrightarrow g \in B_i, \quad i = 1, \dots, q \text{ and } g \in G.$$

Let $\eta(\cdot): G \rightarrow \{1, \dots, q\}$ be a uniformly recurrent point of $G \curvearrowright_\varphi X$ by Zorn's lemma. Since the q clopen blocks

$$[i]_{\mathbf{o}} = \{x(\cdot): G \rightarrow \{1, 2, \dots, q\} \mid x(\mathbf{o}) = i\}, \quad i = 1, 2, \dots, q$$

form an open cover of X where \mathbf{o} is the zero element of $(G, +)$, hence some block $[j]_{\mathbf{o}}$ is an open neighborhood of $\eta(\cdot)$. Write $S = \{g \in G: \varphi(g, \xi(\cdot)) \in [j]_{\mathbf{o}}\}$ which is nonempty for $B_j \neq \emptyset$. Then S is a weak central set of $(G, +)$ by Def. 3.6 and moreover $S = B_j$. This proves Lemma 3.9. \square

It should be noticed that although $\{1, \dots, q\}^G$ is a compact Hausdorff space, yet it is not necessarily a metric space for G is possibly uncountable. Because of this reason, we cannot employ the classical pointwise topological multiple recurrence theorem ([17, Theorem 1.4] and [15, Theorem 2.6]) or the measure-theoretic multiple recurrence theorem of Furstenberg ([15]) that are only for dynamical systems over compact metric spaces or standard Borel spaces.

⁴If G is uncountable and equipped with a locally compact second countable Hausdorff topology, then $\varphi(g, x)$ is not jointly continuous with respect to $g \in G$ and $x \in X$, even not Borel measurable. Here the discrete topology of G enables us to employ Furstenberg's correspondence principle.

3.5. Van der Waerden-type theorems

Now we are able to readily prove the van der Waerden theorem of semimodules over discrete semirings.

Proof of Theorem 1.1. The statement of Theorem 1.1 follows at once from Lemma 3.8 together with Lemma 3.9. \square

Inspired by Furstenberg's concept—VDW-set in \mathbb{Z}^m [15, §2.4], we now introduce and strengthen this notion for semimodules.

Definition 3.10. Given any semimodule $(G, +)$ over a discrete semiring $(R, +, \cdot)$, we say that a subset $B \subseteq G$ is a *van der Waerden-set* (for short *vdW-set*) if for every finite set $F \subseteq G$ we can find a syndetic subset D_F of $(R, +)$ such that for each $d \in D_F$ there is some $a \in G$ with $a + dF \subseteq B$.

The following is another consequence of Theorem 1.2.

Theorem 3.11. *If S is a syndetic subset of a semimodule $(G, +)$ over a discrete semiring $(R, +, \cdot)$, then S is a vdW-set.*

Proof. (Since for a general semigroup $(G, +)$ and a finite subset K of G

$$(K + a) \cap S \neq \emptyset \quad \forall a \in G \not\Rightarrow G = \bigcup_{k \in K} (S - k), \quad \text{for here '}' - \text{' makes no sense [9]!}$$

the proof idea of [15, Proposition 2.8] for $R = \mathbb{Z}^m$ by van der Waerden theorem is invalid here. Differently we will prove this result by using our multiple recurrence theorem and Furstenberg's correspondence principle.)

Let $X = \{0, 1\}^G = \{x_\cdot : G \rightarrow \{0, 1\}\}$ with the product topology and consider the shift dynamical system $\varphi : G \times X \rightarrow X$, or write $G \curvearrowright_\varphi X$, given by

$$g(x_\cdot) = x_{\cdot+g} \quad \forall g \in G \text{ and } x_\cdot \in X.$$

Define a point ξ_\cdot in X as follows

$$\xi_g = 1 \Leftrightarrow g \in S, \quad \forall g \in G.$$

Since the orbit closure $cls_X G[\xi_\cdot]$ is G -invariant, we can find a minimal point $y_\cdot \in cls_X G[\xi_\cdot]$ for $G \curvearrowright_\varphi X$; i.e., $cls_X G[y_\cdot]$ is minimal for $G \curvearrowright_\varphi X$ such that $y_\cdot \in cls_X G[\xi_\cdot]$. As S is “syndetic” in G associated to some finite subset, say $K = \{g_1, \dots, g_k\} \subseteq G$, it follows that there exists some element $\hat{g} \in G$ such that

$$\xi_{g_1+\hat{g}} = y_{g_1}, \dots, \xi_{g_k+\hat{g}} = y_{g_k} \quad \text{and then } 1 \in \{y_{g_1}, \dots, y_{g_k}\}$$

Without loss of generality, assume $y_{g'} = 1$. Since the cylinder set

$$U = \{x_\cdot \in X \mid x_{g'} = 1\}$$

is an open neighborhood of y_\cdot in X , then by Theorem 1.2, it follows that for any finite set $F \subseteq G$ there exists a syndetic set $D_F \subseteq (R, +)$ such that for any $d \in D_F$ one can choose some point $z_\cdot \in U \cap cls_X G[\xi_\cdot]$ with $z_{|dF} \equiv 1$. This implies that there is some $a \in G$ so that $\xi_{|a+dF} \equiv 1$. Hence $a + dF \subseteq S$.

This thus proves Theorem 3.11. \square

We now present an application of our van der Waerden-type result Theorem 1.1.

Theorem 3.12. *Let Λ be a compact metric space, $(G, +)$ a semimodule over a discrete semiring $(R, +, \cdot)$, and let $f: G \rightarrow \Lambda$ be an arbitrary function. Then for any $\varepsilon > 0$ and finite set $F \subseteq G$, we can find a syndetic subset D of $(R, +)$ such that for any $d \in D$ there exists an $a \in G$ so that $f(a + dF)$ is a set of diameter less than ε in Λ .*

Proof. This follows from Theorem 1.1 by an argument same as that of [15, Theorem 2.9] for $R = \mathbb{N}^m$. So we omit the details here. \square

Theorem 3.12 implies that if $f: G \rightarrow \mathbb{R}$ is a bounded function on a semigroup G and $\varepsilon > 0$, then there will be three elements in “arithmetic progression” $a, a + h, a + 2h$ in G such that $|f(a) - f(a + h)| < \varepsilon$ and $|f(a + h) - f(a + 2h)| < \varepsilon$.

Corollary. *Let X be an arbitrary space and T_1, \dots, T_l commuting transformations of X to itself, and let ϕ_1, \dots, ϕ_l be any functions from X to the unit circle \mathbb{T} : $|\phi_i(x)| = 1$. Then for any $\varepsilon > 0$ and any $m \in \mathbb{N}$, we can find a syndetic subset D of \mathbb{Z}_+ such that to any $n \in D$ there is an $x_n \in X$ to satisfy the inequalities:*

$$\left| \phi_i(T_1^{k_1 n} T_2^{k_2 n} \dots T_l^{k_l n} x_n)^{k_0 n} - 1 \right| < \varepsilon \quad \forall i = 1, \dots, l$$

for all $(k_0, k_1, \dots, k_l) \in \mathbb{Z}_+^{l+1}$ with $0 \leq k_j \leq m$ for each $j = 0, 1, \dots, l$.

Proof. Given any point $x_0 \in X$, we now define the vector-valued function $\Phi: \mathbb{Z}_+^{l+1} \rightarrow \mathbb{T}^l$ by setting

$$\Phi(n_0, n_1, \dots, n_l) = \begin{pmatrix} \phi_1(T_1^{n_1} T_2^{n_2} \dots T_l^{n_l} x_0)^{n_0} \\ \phi_2(T_1^{n_1} T_2^{n_2} \dots T_l^{n_l} x_0)^{n_0} \\ \vdots \\ \phi_l(T_1^{n_1} T_2^{n_2} \dots T_l^{n_l} x_0)^{n_0} \end{pmatrix} \quad \forall (n_0, n_1, \dots, n_l) \in \mathbb{Z}_+^{l+1}.$$

Choosing a metric on \mathbb{T}^l by $\|(z_1, \dots, z_l) - (y_1, \dots, y_l)\| = \max_{1 \leq i \leq l} |z_i - y_i|$, we proceed by Theorem 3.12 with $(G, +) = (\mathbb{Z}_+^{l+1}, +)$ over the semiring $(\mathbb{Z}_+, +, \cdot)$ to find a “syndetic” subset D of \mathbb{Z}_+ associated to $\varepsilon > 0$ and the $(l + 1)$ -dimensional cube

$$F = \{(k_0, k_1, \dots, k_l) \in \mathbb{Z}_+^{l+1} \mid 0 \leq k_j \leq m, 0 \leq j \leq l\}$$

such that for any $n \in D$, one can find some element $a = (n_0, n_1, \dots, n_l) \in \mathbb{Z}_+^{l+1}$ with

$$\text{diam}(\Phi(a + nF)) < \varepsilon.$$

If we now set

$$x_n = T_1^{n_1} T_2^{n_2} \dots T_l^{n_l} x_0$$

and compare the values of Φ at the vertices of the homothetic copy $a + nF$ of the cube F , we can see that for any $i = 1, 2, \dots, l$ and all $(k_0, k_1, \dots, k_l) \in F$,

$$\left| \phi_i(T_1^{k_1 n} \dots T_l^{k_l n} x_n)^{n_0} - \phi_i(T_1^{k_1 n} \dots T_l^{k_l n} x_n)^{n_0 + k_0 n} \right| < \varepsilon \quad \text{or} \quad \left| \phi_i(T_1^{k_1 n} \dots T_l^{k_l n} x_n)^{k_0 n} - 1 \right| < \varepsilon.$$

This therefore proves the corollary of Theorem 3.12. \square

It should be noted here that this result is just a strengthen of a theorem of Furstenberg [15, Theorem 2.13].

Now we let $(G, +)$ be a semimodule over a discrete $*$ -semiring $(R, +, \cdot)$. The following variation of Theorem 1.1 is just the “finitary formulation” in which one considers partitions of large sets (but finite if G is countable).

Theorem 3.13. *Let $G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$ with $G = \bigcup_{n \geq 1} G_n$; and let F be a finite subset of G and $q \in \mathbb{N}$. Then there exists a number $N = N(q, F)$ such that whenever $n \geq N$ and $G_n = B_1 \cup \cdots \cup B_q$ is a partition into q sets, one of these B_j contains a homothetic copy of F , $a + dF$, where $a \in G$ and $d \in R$ with $d \neq 0$.*

Proof. We think of the partition $G_n = B_1 \cup \cdots \cup B_q$ as a function ξ_n from G_n to $\{1, 2, \dots, q\}$ defined by $\xi_n(g) = j \Leftrightarrow g \in B_j$ for $j = 1, \dots, q$, for any $n \geq 1$. Suppose with $n \rightarrow +\infty$ we can find partitions for which no homothetic copy of F is contained in any cell B_j . Consider the corresponding function from G_n to $\{1, 2, \dots, q\}$ and extend it arbitrarily onto G to obtain a point $\xi_n \in \{1, 2, \dots, q\}^G$. Take any limit point of $\{\xi_n\}$, say ξ , and apply Theorem 1.1 to the corresponding partition

$$G = \{\xi = 1\} \cup \cdots \cup \{\xi = q\}.$$

It follows that ξ is constant on some homothetic copy $a + dF$ where $a \in G$ and $d \in R$ with $d \neq 0$. This set $a + dF$ is contained in G_n as soon as n is large, and moreover ξ_n agrees with ξ on $a + dF$ for some large n . But this clearly leads to a contradiction that proves the theorem. \square

This theorem generalizes obviously the classical result [15, Theorem 2.10] for \mathbb{N}^m . It is clear that since G does not need to be countable here, this version does not imply our previous formulation Theorem 1.1 in general.

3.6. An open problem

Finally we conclude our discussion with the following open question closely related to our topic:

Conjecture 3.14 (Schur-Brauer version of van der Waerden’s theorem). *Let $G = B_1 \cup \cdots \cup B_q$ be any finite partition of a semimodule $(G, +)$ over a semiring $(R, +, \cdot)$. One of the sets B_j has the property that if F is any finite subset of R , then there are elements $a \in G$ and $b \in B_j$ with $b \neq 0$ such that $a + Fb \subseteq B_j$.*

Notice here that ‘ $F \subseteq R$ ’ in Conjecture 3.14, but not ‘ $F \subseteq G$ ’ as in Theorem 1.1 there. If $G = \mathbb{Z}_+$, this is just the Schur-Brauer theorem.

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